Introduction to latin bitrades

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Nonassociative triples

Let $Q(*)$ be a quasigroup or order n. Denote by $s = s(Q)$ the size of $\{(x, y, z) \in Q^3; x * (y * z) \neq (x * y) * z\}$.

- Then there exists an integer t such that

$$4tn - 2t^2 - 24t \leq s \leq 4tn, \text{ and} \tag{$*$}$$

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Inequalities ($*$) powerful if $t << n$. Then $s \leq 4tn$ gives a good upper bound on s. If t is small, then s is small, and the implication ($\dagger$) gives a lower bound for s.

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To get a lower bound for the least possible value of $s \geq 1$ we need to know whether the least possible value of $t \geq 1$ satisfies $4t < 3n/32$. Possible for $n \geq 1171$. (If $n \geq 168$ is even, then $s = 16n - 64 = 4t - 64$).

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- The answer is a qualified **yes**.
- Trades were also discovered by studying *critical sets* of a latin square (the minimum partial squares with unique completion—i.e. partial squares that intersect all trades).

Latin bitrades

A *latin bitrade* $(K^*, K^\triangle)$ is standardly defined as a pair of two partial latin squares in which the same cells are occupied but never by the same symbol. They have to be *row balanced* and *column balanced* (the set of symbols in a given row or column is the same in both partial latin squares).

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1	2	3
4		1
2	3	4

2	3	1
1		4
4	2	3

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$$\begin{array}{c|ccc} * & 1 & 2 & 3 \\ \hline 1 & 1 & 2 & 3 \\ 2 & 4 & & 1 \\ 3 & 2 & 3 & 4 \end{array} \qquad \begin{array}{c|ccc} \triangle & 1 & 2 & 3 \\ \hline 1 & 2 & 3 & 1 \\ 2 & 1 & & 4 \\ 3 & 4 & 2 & 3 \end{array}$$

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$\triangle$	1	2	3
1	2	3	1
2	1		4
3	4	2	3

Consider K^*, $K^\triangle$ as sets of triples (row,column,symbol). $K^* = \{(1,1,1),(1,2,2),\dots\}$, $K^\triangle = \{(1,1,2),(1,2,3),\dots\}$.

Bitrades as triple sets

Let K^* and $K^\triangle$ be sets of triples. They form a latin bitrade iff for every $\alpha = (a_1, a_2, a_3) \in K^*$ and $i \in \{1,2,3\}$ there exists a unique $\beta = (b_1, b_2, b_3) \in K^\triangle$ with $a_i \neq b_i$ and $a_j = b_j$ for $j \neq i$, $j \in \{1,2,3\}$. (α and β agree in exactly two coordinates.)
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A latin bitrade is called *separated* if it yields a surface. Non-separated bitrades have a row (or a column, or a symbol) that can be divided into two rows (or two columns, or two symbols).

A non-separated bitrade

1	2	3	4
2	3	4	1
3		1	

2	3	4	1
3	2	1	4
1		3	

A non-separated bitrade

1	2	3	4			2	3	4	1
2	3	4	1			3	2	1	4
3		1				1		3	

can have its middle row divided:

1	2	3	4			2	3	4	1
2	3					3	2		
		4	1					1	4
3		1				1		3	

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For the group distance problem the *spherical* bitrades are the most relevant. They are separated by definition.

Bitrades, surfaces and permutations

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The genus g of the surface can be computed as

$$\text{size} + 2(1 - g) = \text{order},$$

where the order is defined as $r + c + s$ (the aggragated number of rows, columns and symbols) and the size is the number of the cells (all triangles of one colour).

3-homogeneous latin bitrades

Let every row and every column contain three cells and let every symbol occur 3 times. Then
size $= 3r = 3c = 3s = r + c + s =$ order, $g = 1$ (a torus).

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$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ 1 & B/W & B/W & B/W & B/W & 3 \\ 2 & B/W & B/W & B/W & B/W & 1 \\ 3 & B/W & B/W & B/W & B/W & 2 \\ & 1 & 2 & 3 & 4 & \end{array}$$

Each B is a black triangle. The vertices of the leftmost top B are (in the order of colours R, C, S) $((1,1),(1,2),(2,1))$. The vertices of the righmost middle B are $((1,1),(3,4),(2,4))$.

Torus expressed via tables

The black triangles of the example yield a partial operation

$*$	$(1,2)$	$(2,3)$	$(3,1)$	$(3,4)$
$(1,1)$	$(2,1)$		$(3,2)$	$(2,4)$
$(2,2)$	$(1,3)$	$(3,2)$	$(2,1)$	
$(1,4)$		$(1,3)$	$(2,4)$	$(2,1)$
$(3,3)$	$(3,2)$	$(2,4)$		$(1,3)$

The white triangles give

$\triangle$	$(1,2)$	$(2,3)$	$(3,1)$	$(3,4)$
$(1,1)$	$(3,2)$		$(2,4)$	$(2,1)$
$(2,2)$	$(2,1)$	$(1,3)$	$(3,2)$	
$(1,4)$		$(2,4)$	$(2,1)$	$(1,3)$
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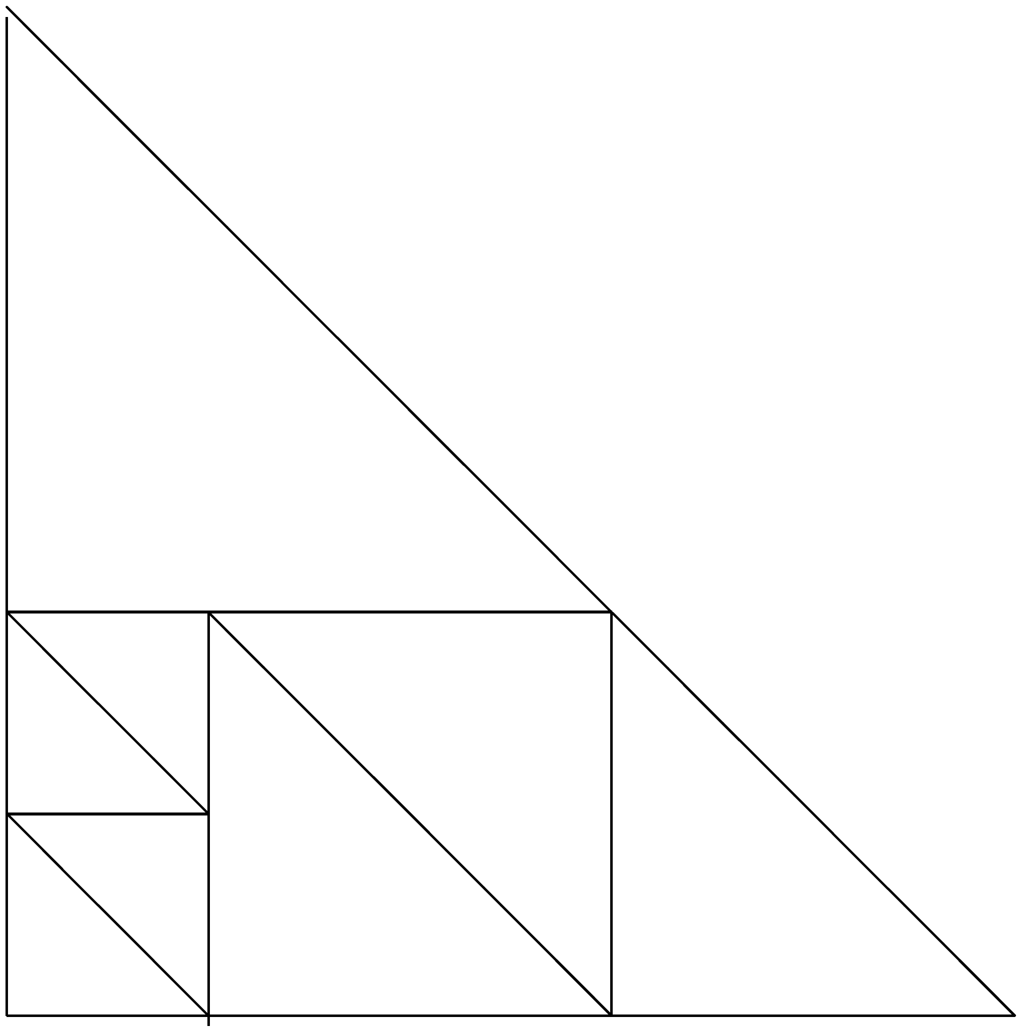
The converse belongs to Cavenagh and Lisoněk (2008) and is based upon a classical result by Heawood (1898) that a spherical triangulations is vertex 3-colourable if and only if all vertices are of even degree. (The colours are the rows, columns and symbols.)

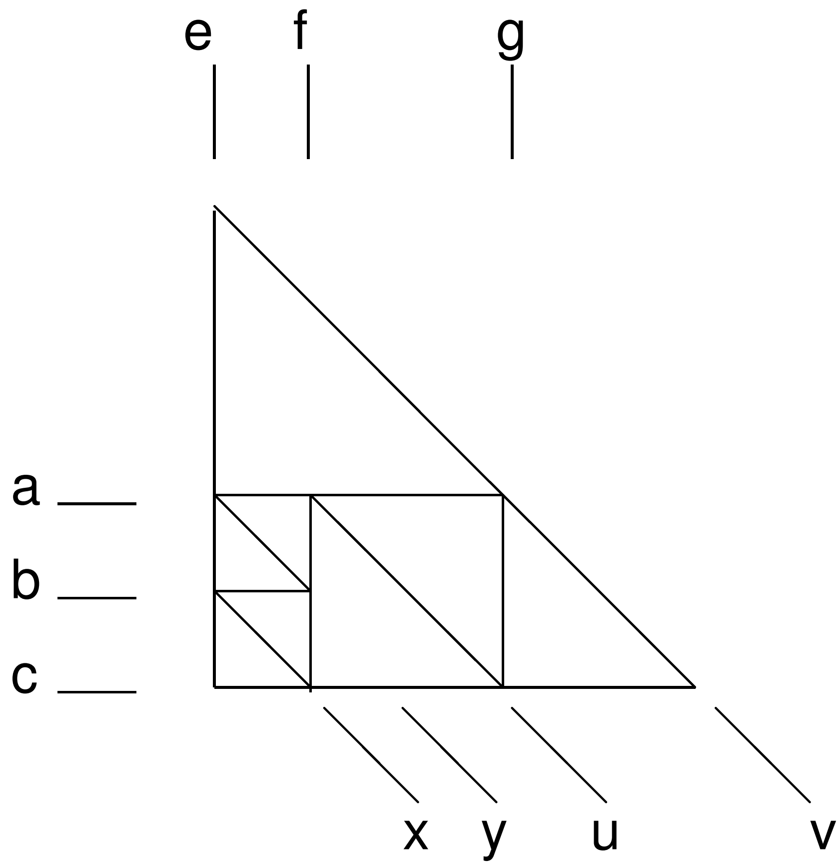
Deriving trades from dissections

Consider a dissection of an equilateral triangle, say



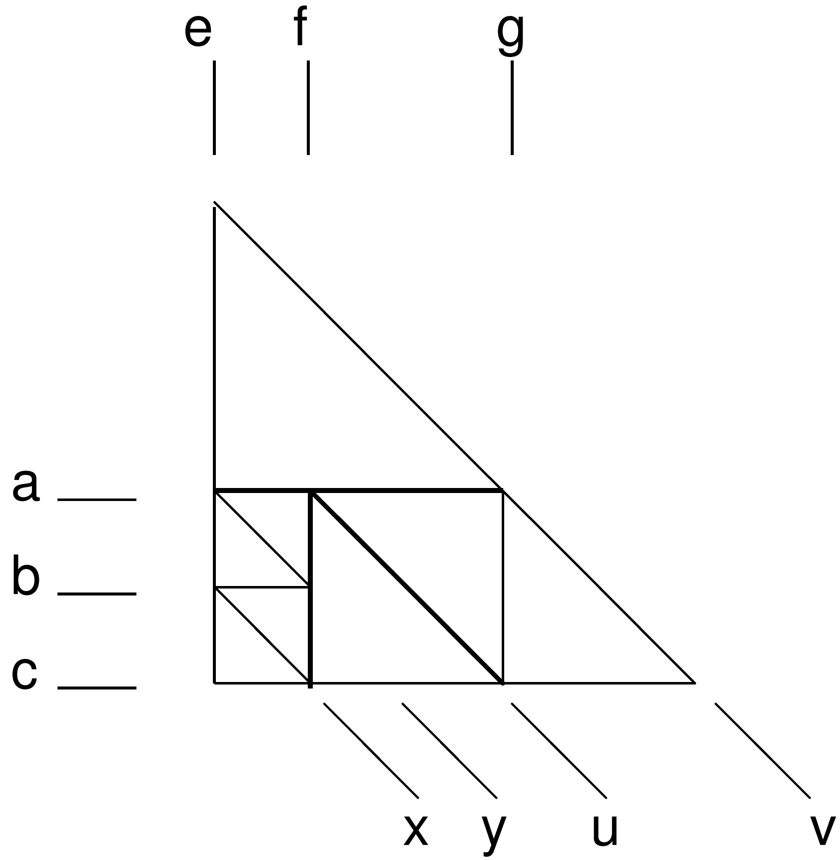
Deriving trades from dissections

The same dissection with named segments:



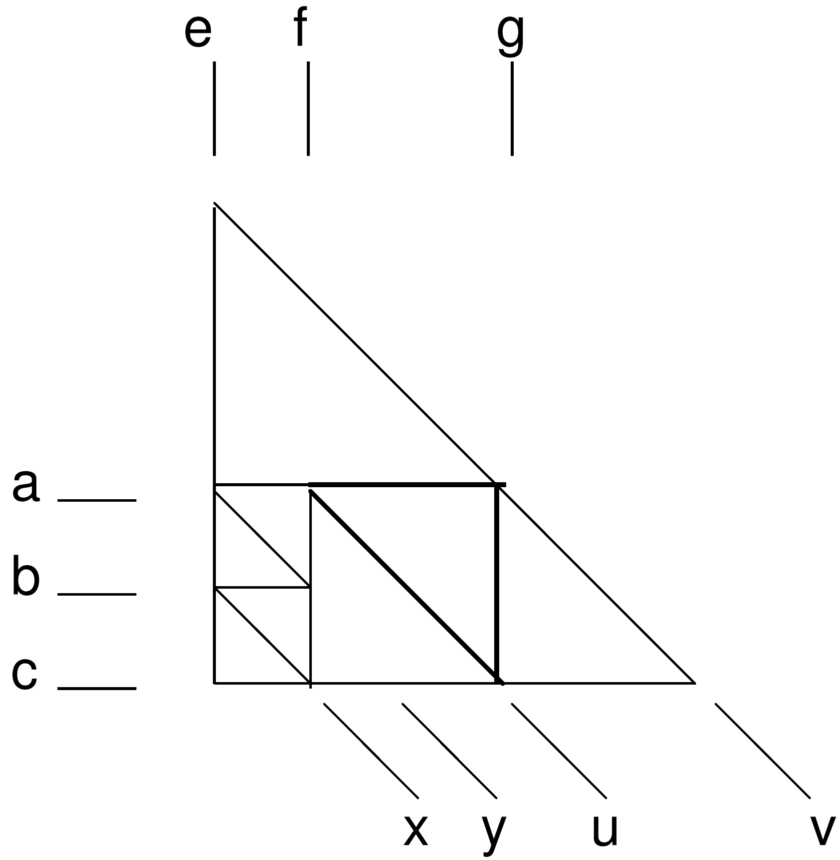
Deriving trades from dissections

The $*$ operation describes intersections of segments. For example $a * f = u$. A special case: $c * e = v$.



Deriving trades from dissections

The $\triangle$ operation describes the dissecting triangles. For example $a \triangle g = u$, i.e. $(a, g, u) \in K^{\triangle}$.



From bitrades to dissections

The example gives a spherical bitrade

$*$	e	f	g
a	y	u	v
b	x	y	
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a	v	y	u
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If we get the bitrade and wish to derive a dissection, we must first choose a triple $(c, e, v) \in K^*$ that determines the outer segments. Numerical values are attributed to $a, b, c, \ldots$ as follows:

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From latin bitrades to dissections

A *pointed latin bitrade* (K, a) is a bitrade $K = (K^*, K^\triangle)$ with $a = (a_1, a_2, a_3) \in K^*$. With such a bitrade associate a set of equations $\mathrm{Eq}(T, a)$ which includes equations $a_1 = 0$, $a_2 = 0$, $a_3 = 1$ and $b_1 + b_2 = b_3$ for every $b = (b_1, b_2, b_3) \in K^*$, $b \neq a$. Assume that K is spherical.

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Denote by Σ the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$. For every $b = (b_1, b_2, b_3) \in K^*$, $b \neq a$, let $P(b, a)$ be the point (β_2, β_1), where β_i is the solution to b_i.

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For $c = (c_1, c_2, c_3) \in K^\triangle$ let $\gamma_1 = (b_1, c_2, c_3)$, $\gamma_2 = (c_1, b_2, c_3)$ and $\gamma_3 = (c_1, c_2, b_3)$ be elements of K^*. Denote by $\Delta(c, a)$ the triangle with vertices $P(\gamma_1, a)$, $P(\gamma_2, a)$ and $P(\gamma_3, a)$.

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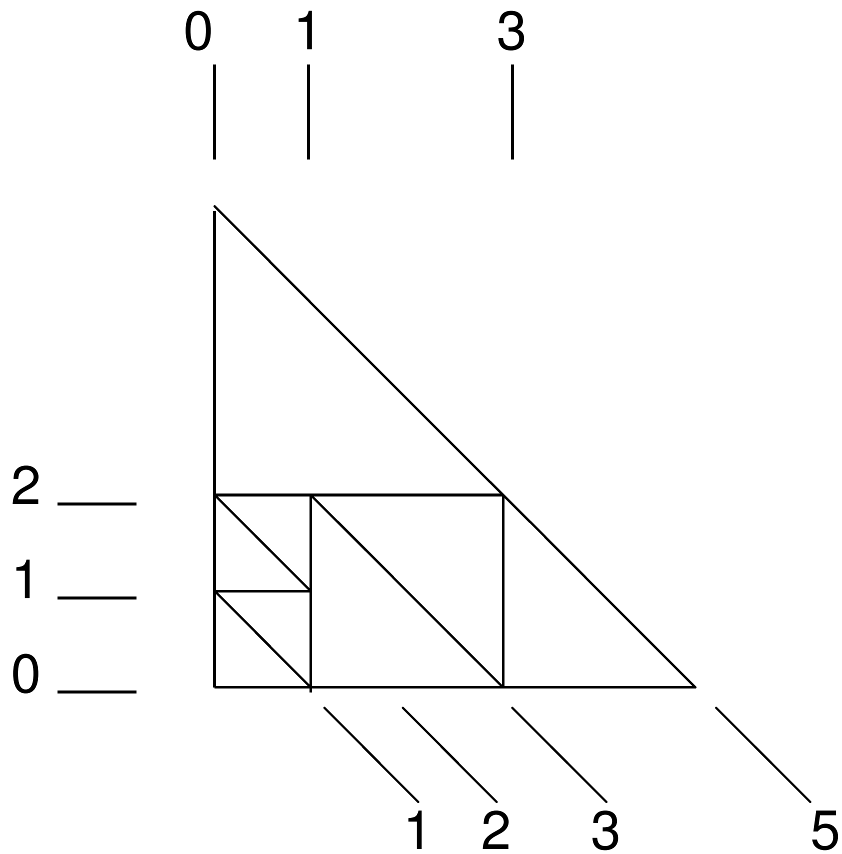
Fact 3 The triangle Σ is dissected by the set of all $\Delta(c, a)$, $c \in K^\triangle$ that do not degenerate.

Dissections and counting modulo

Every dissection can be adjusted to integer coordinates. Let n be the size of the dissected triangle. Then $b_1 * b_2 = b_3$ implies $b_1 + b_2 = b_3 \mod n$. This means that $K(*)$ can be embedded into $\mathbb{Z}_n(+)$. Example:



Embedding a spherical latin bitrade

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By using $a' \in K^*$ in place of a, we obtain n' and $\varphi' = (\varphi'_1, \varphi'_2, \varphi'_3)$. Together φ and φ' send $K(*)$ to $\mathbb{Z}_n \times \mathbb{Z}_{n'}$.

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Functors

Let $K = (K(*), K(\triangle))$ be a latin bitrade. Suppose that $K = R \cup C \cup S$, where the sets R, C and S are pairwise disjoint (rows, columns, symbols).

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- However, there exist toroidal trades such that $K(*)$ embeds into $\mathbf{H}(K(*))$, while $\mathbf{H}(K(\triangle))$ is trivial.

Generating latin bitrades

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- There have been described several constructive methods how to blow up the genus, and thus to obtain latin bitrades of all genera when starting from spherical trades. None of them seems to have been really implemented.
- The *plantri* program can be used to generate graphs up to the vertex size 50, which means the trade size up to 25.

Trades and group distances

- The question about $\mathrm{gdist}(n) = \min \mathrm{dist}(G, Q)$, Q a quasigroup, G group of order n can be reformulated by asking about the trade $K = (K(*), K(\vartriangle))$ of the least possible size m such that $K(*)$ embeds into a group of order n.

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- There is no formal proof that every such trade K is spherical. However, it is plausible to assume that.
- There is no formal proof that every such spherical trade K may be embedded into a cyclic group. However, this is plausible to assume since it seems natural to expect that $K(*)$ embeds into $\mathbb{Z}_p$, where p is the least prime dividing n.

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- There have computed all $t(n)$ with $t(n) \leq 23$. The greatest such n is equal to 433. For n higher than 30 the only way seems to be to use estimates.
- Besides $t(n)$ let us consider also $\hat{t}(n)$ which refers to spherical latin bitrades that embed to $\mathbb{Z}_n$, but not to $\mathbb{Z}_d$, d a proper divisor of n (integral dissections of size n with the $\gcd$ of dissecting triangles equal to 1).

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- **Conjecture:** $\mathrm{spb}(n) - 1 \leq \hat{t}(n) \leq \mathrm{spb}(n)$. The *spiral bound* $\mathrm{spb}(n)$ is defined by

$$a_{\mathrm{spb}(n)-1} < n \leq a_{\mathrm{spb}(n)}, \quad \text{where}$$
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- That would suggest the “right” constant to be ~ 3.56.